THE UNIVERSITY OF ALBERTA

INDEPENDENT SETS OF POSTULATES FOR BOOLEAN ALGEBRAS, DISTRIBUTIVE LATTICES, AND GENERAL LATTICES

by

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A THESIS

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ABSTRACT

A.A. Grau [6] defined a ternary Boolean algebra in terms of one ternary operation and one unary operation by five postulates. In this thesis it is shown that only three of the five postulates are sufficient to define a ternary Boolean algebra. The three postulates are shown to be independent. Other definitions of a ternary Boolean algebra in terms of one ternary operation and one unary operation by means of three independent postulates are also given. Either by combining or permuting the most familiar axioms or by both, the number of postulates in each case is reduced to a minimum of two independent axioms.

A number of sets of independent axioms satisfying Birkhoff's problem 64 [1] are obtained for a bounded distributive lattice. Each system is transformed into a system of two independent postulates in many different ways.

Lattices and semilattices are defined without the law of commutativity. 112 sets of four independent axioms and 1112 sets of three independent axioms for lattices and two sets of two independent axioms for semilattices are constructed.

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CHAPTER I

1. INTRODUCTION

Ordinary axiomatizations of different kinds of lattices including Boolean algebras may be divided into two classes according as their original undefined concepts are operations or functions such as product, sum and complement employed along with a relation of equality or identity or a relation of inclusion, while of course, combinations of the two are also possible. Examples of the operational type have been much more numerous. But some interest has been attached to definitions of special kinds of lattices and Boolean algebras in terms of a ternary operation. Ternary operations are defined in groupoids and groups. But the ternary operation we are using here is the one introduced by Grau [6]. The tendency has been for the number of postulates to be much smaller. This thesis shows a comparable economy as regards the number of postulates. That is, the number of axioms finally employed will be two. The first step will be to give a set of three postulates of more familiar appearance. Then a reduction by one will follow. It does not seem to be possible to reduce the number of axioms to one in each case.

Chapter II of this thesis deals with sets of axioms for a ternary Boolean algebra. A.A. Grau [6] defined a ternary Boolean algebra in terms of a ternary operation and a unary operation by means of five axioms. In section 5 of the thesis, it is shown that only three of the five axioms define a ternary Boolean algebra. It is also shown that these three postulates are independent. Other sets of three independent postulates for a ternary Boolean algebra are also obtained. In section 6, some sets of two independent postulates for a ternary Boolean algebra are discussed.

Chapter III is devoted to the study of sets of axioms for bounded distributive lattices. In section 7, G. Birkhoffs problem 64, posed on p. 138 of the 1948 edition of his book on lattice theory, is mentioned. In section 8, 297 new solutions of this problem, 246 of which are sets of three independent axioms and the remaining are sets of four independent axioms, are obtained. In section 9, further sets of two independent postulates for a bounded distributive lattice are stated.

In chapter IV, postulates for arbitrary semilattices and lattices, without the law of commutativity, are investigated. D.H. Potts [9] stated a set of two axioms for semilattices in six variables. In section 11 two sets of two independent axioms without the law of commutativity in only three variables are obtained. In the same section associative and absorption laws for the lattice operations of join and meet are combined in 8192 different ways instead of 3200 ways as in the case of A. Petcu [8], and it is established that 1128 sets of three independent identities and 112 sets of four independent identities in these operations define a lattice. On the other hand, it is also shown that none of some 48 further sets of four identities in the same operations is a set of lattice axioms.

2. DEFINITIONS

Definition 2.1. A ternary operation on a set A is a mapping of $A \times A \times A$ into A.

Definition 2.2. A ternary operation (abc) on A is said to be completely commutative iff it is invariant under all permutations of a, b and c.

3. NOTATIONS

'A' always stands for a set of elements and the lower case letters a,b,c,d,e,x and y denote the elements of A. The letters O and I denote the lower and upper bound elements of a bounded lattice and 0 and 1 are the zero and unit of a Boolean algebra. The complement of an element 'a' of a lattice is denoted a' if this complement exists and is unique. The binary lattice operations of join and meet are denoted by the usual symbols $\vee$ and $\wedge$, respectively.

CHAPTER II

SETS OF INDEPENDENT AXIOMS FOR A BOOLEAN ALGEBRA

4. INTRODUCTION

A.A. Grau [6] defined a ternary Boolean algebra by a system of five postulates as follows:

Let $\mathcal{A}$ be an algebraic system consisting of a set A, a ternary operation (abc) and a unary operation a' which satisfy the following identities:

(4.1) $$(de(abc)) = ((dea)b(dec))$$

(4.2) $$(baa) = a$$

(4.3) $$(aab) = a$$

(4.4) $$(abb') = a$$

(4.5) $$(b'ba) = a.$$

Then $\mathcal{A}$ is called a ternary Boolean algebra.

He also proved that if two binary operations, $\wedge$ and $\vee$, are defined by

$$
(4.6) \qquad \begin{cases} a \wedge b = (a0b) \\ \\ a \vee b = (a0'b) \end{cases}
$$

where 0 is a fixed element of A, then $(A,\wedge,\vee)$ is a Boolean algebra, i.e., a complemented distributive lattice. Here 0 and $0'$ are the bound elements of this lattice, a' is the complement of a with respect to this lattice for every a, and

$$
(4.7) \qquad (abc) = (a\wedge b) \vee (b\wedge c) \vee (c\wedge a)
$$

holds identically.

Conversely, if $(A,\wedge,\vee)$ is any complemented distributive lattice and a ternary operation (abc) is defined by (4.7) then (4.1) to (4.6) hold identically, whence A together with this ternary operation and the operation $a \to a'$ is a ternary Boolean algebra.

Later on, by suitable permutations of (4.1) and (4.4), R. Croisot [5] replaced the system of five postulates for a ternary Boolean algebra by this system of two postulates:

$$
(4.1.1) \qquad (de(bac)) = (b(edc)(dea))
$$

(4.8) $$(bab') = a.$$

He also showed that the axioms (4.1.1) and (4.8) are independent.

In this thesis it is shown that the axioms (4.3) and (4.5) can be derived from the other three postulates, (4.1), (4.2) and (4.4). In other words, three postulates, (4.1), (4.2) and (4.4), out of the five postulates of Grau, are sufficient to define a ternary Boolean algebra. It is also shown that these three postulates are independent.

A number of new systems of independent axioms consisting of three identities are also obtained in section five.

In section six, the axiom systems consisting of three identities which occur in section five are reduced to systems of two independent postulates in many different ways.

In the following by a "Boolean algebra" is meant a ternary Boolean algebra in the sense of A.A. Grau [6].

5. SETS OF THREE POSTULATES

In this section, sets of postulates which consist of only three identities are given.

THEOREM 5.1. Let $\mathcal{A}$ be any algebraic system consisting of a set A, a ternary operation (abc) and a unary operation a'

such that

$$(4.1) \qquad (de(abc)) = ((dea)b(dec))$$

$$(4.2) \qquad (baa) = a$$

$$(4.4) \qquad (abb') = a$$

identically. Then A is a Boolean algebra.

Proof. It is sufficient to show that (4.1), (4.2) and (4.4) imply (4.3) and (4.5).

By (4.4), (4.1), (4.4), (4.2), we obtain

$$(5.1.1) \qquad \begin{aligned}(aab) &= (aa(baa')) \\ &= ((aab)a(aaa')) \\ &= ((aab)aa) = a.\end{aligned}$$

By (4.4) (4.2), (4.1), (5.1.1), (4.4), we get

$$(5.1.2) \qquad \begin{aligned}(ab'b) &= ((abb')b'(abb)) \\ &= (ab(b'b'b)) \\ &= (abb') = a.\end{aligned}$$

By (4.2), (4.1), (4.2), (4.1), (5.1.2) (4.2), (5.1.1),

$$(5.1.3)\qquad \begin{aligned}(bab') &= (ba(abb'))\\ &= ((baa)b'(bab')\\ &= (ab'(bab'))\\ &= ((ab'b)a(ab'b'))\\ &= (aab') = a.\end{aligned}$$

By (4.2) (4.4), (4.1), (5.1.3),

$$(5.1.4)\qquad \begin{aligned}(abc) &= ((caa)b(caa'))\\ &= (ca(aba'))\\ &= (cab).\end{aligned}$$

It follows from (4.2) and (5.1.4) that

$$(5.1.5)\qquad (aba) = a.$$

By (5.1.3), (4.1), (5.1.5) (5.1.3),

$$(5.1.6)\qquad \begin{aligned}(abc) &= (ab(aca'))\\ &= ((aba)c(aba'))\\ &= (acb).\end{aligned}$$

Now it is clear from (5.1.4) and (5.1.6) that (abc) is invariant under all permutations of a, b and c, i.e., completely commutative.

This together with (4.4) implies that

(5.1.7) $(b'ba) = (abb') = a.$

Now (5.1.1) includes (4.3) and (5.1.7) includes (4.5). Hence (4.1), (4.2) and (4.4) imply (4.3) and (4.5) as we wished to prove.

It will now be shown that the axioms (4.1), (4.2) and (4.4) are independent. That is, it will be shown that, for $j = 1,2,4$, it is possible to find an algebra with a ternary operation (abc) and a unary operation a' such that (4.k) holds for $k \in \{1,2,4\} - \{j\}$ while (4.j) does not hold.

A_1. Consider the set $B = \{u,v\}$, $u \neq v$ and define a ternary operation on it by $(abc) = u$ or v according as atleast two of a,b,c are u or v and let $a' = a$ for all $a \in B$.

Then, with the symbols a,b,c,d,e denoting any of the elements u,v, (4.1) and (4.2) are satisfied but (4.4) does not hold identically.

A_2. Assume again the set $\{u,v\}$ and define on it a ternary operation by $(abc) = a \circ c$, where $\circ$ is the binary operation defined by $v \circ u = u \circ v = v \circ v = v$, and $u \circ u = u$, and a unary operation by $u' = v' = u$.

Then (4.1) and (4.4) are evidently satisfied while (4.2) is not valid.

A_3. Lastly, consider the set $B = \{u,v,w,z\}$ of four different elements u,v,w,z, and define thereupon a two operation algebra in the following way: $u' = v$, $v' = u$, $z' = w$, $w' = z$, and $(abc) = c$ if $c = b$, $(abc) = a$ if $c = b'$, $(abc) = u$ otherwise.

Then (4.2) and (4.4) are true but (4.1) is false; for if $a = v$, $b = d = z$, $c = w$, $e = u$, then $(de(abc)) = (zu(vzw)) = (zuv) = z$, while

$$\begin{aligned}((dea)b(dec)) &= ((zuv)z(zuw)) \\ &= (zzu) = u.\end{aligned}$$

Thus the axioms (4.1), (4.2) and (4.4) are mutually independent.

It will now be shown how a Boolean algebra can be defined by (4.2), (4.4) and a permutation of (4.1).

THEOREM 5.2. Let

(4.2) $$(baa) = a$$

(4.4) $$(abb') = a$$

and

(4.1.2) $$(de(abc)) = (a(deb)(dec))$$

identically. Then

$$(abc) = (cba) \tag{5.2.1}$$

$$(aab) = a \tag{5.2.2}$$

$$(b'ab) = a \tag{5.2.3}$$

$$(bab') = a \tag{5.2.4}$$

and

$$(abc) = (bca). \tag{5.2.5}$$

Proof. By (4.4), (4.1.2), (4.2) (4.4), we get

$$\begin{aligned}(abc) &= (ab(cbb')) \\ &= (c(abb)(abb')) \\ &= (cba).\end{aligned}$$

Hence (5.2.1) holds.

(4.2) and (5.2.1) imply (5.2.2). By (4.4) (4.2), (4.1.2), (5.2.2), (4.4),

$$(b'ab) = (b'(abb')(abb))$$
$$= (ab(b'b'b))$$
$$= (abb') = a.$$

Hence (5.2.3) holds.

(5.2.4) now follows from (5.2.3) and (5.2.1). Finally,

$$(abc) = (a(bcc')(bcb'))$$
$$= (bc(ac'b'))$$
$$= (bc(b'c'a))$$
$$= (b'(bcc')(bca))$$
$$= (b'b(bca))$$
$$= (bca)$$

by (4.4) (5.2.4), (4.1.2), (5.2.1), (4.1.2), (4.4), (4.4) (5.2.1). Hence (5.2.5) holds and the theorem is proved.

Now, by (5.2.1) and (5.2.5) (abc) is completely commutative. Therefore (4.1.2) implies (4.1). Thus (4.2), (4.4) and (4.1.2) imply the hypotheses of Theorem 5.1. Hence (4.2), (4.4) and (4.1.2) also define a Boolean algebra.

(4.2), (4.4) and (4.1.2) are independent by A_2, A_1 and A_3.

THEOREM 5.3. Let $\mathcal{A}$ be an algebraic system consisting of a set A, a ternary operation (abc) on A and a unary operation

a' on A. Then $\mathcal{A}$ is a Boolean algebra if and only if the following three identities hold:

(4.2) $$(baa) = a$$

(4.9) $$(b'ab) = a$$

(5.3.1) $$((bde)(aed)c) = (b(edc)(aed)).$$

Proof. First a few identities are proved which will be used in the proof of the theorem. Put $a = d'$ in (5.3.1) and use (4.9) to obtain

(5.3.2) $$((bde)ec) = (b(edc)e).$$

Again, put $b = e'$ in (5.3.2) to get

(5.3.3) $$(dec) = (edc)$$

by (4.9). Now by (4.2), (5.3.2), (5.3.3), (5.3.2), (5.3.3), (5.3.2), (4.2),

(5.3.4) $$\begin{aligned}(abc) &= (a(cbb)c)\\ &= ((abc)cb)\\ &= ((bac)cb)\\ &= (b(cab)c)\end{aligned}$$

$$= ((acb)bc)$$

$$= (a(bcc)b)$$

$$= (acb).$$

Then (5.3.3) and (5.3.4) together imply that (abc) is completely commutative.

Let 0 be an element of A. Let $\wedge$ and $\vee$ be the binary operations on A defined by the identities

$$(4.6) \qquad \begin{cases} a \wedge b = (a0b) \\ a \vee b = (a0'b). \end{cases} \qquad \text{and}$$

It will now be shown that $(A,\wedge,\vee)$ is a lattice. By (4.6), (5.3.3) and (5.3.4), we get

$$(5.3.5) \qquad \begin{cases} a \wedge b = (a0b) = (b0a) = b \wedge a \\ a \vee b = (a0'b) = (b0'a) = b \vee a. \end{cases}$$

Again by (4.6), (5.3.2) to (5.3.4),

$$(a0(b0c)) = ((a0b)0c).$$

Hence

(5.3.6) $$a \wedge (b \wedge c) = (a \wedge b) \wedge c.$$

Similarly,

(5.3.7) $$a \vee (b \vee c) = (a \vee b) \vee c.$$

Besides,

$$\begin{aligned}(a0(a0'b)) &= (b(0'a0)a)\\ &= (baa) = a\end{aligned}$$

by (5.3.3) (5.3.4) (5.3.2), (4.9), (4.2).

Hence

(5.3.8) $$a \wedge (a \vee b) = a.$$

Similarly, we obtain

(5.3.9) $$a \vee (a \wedge b) = a.$$

Thus by (5.3.5), $\wedge$ and $\vee$ are commutative. By (5.3.6) and (5.3.7), they are associative. They satisfy the absorption identities, (5.3.8) and (5.3.9). Hence $(A, \wedge, \vee)$ is a lattice.

Now we shall show that $(A, \wedge, \vee)$ is modular.

If $a \leq c$, then

$$(5.3.10) \qquad \begin{cases} (a0c) = a \wedge c = a & \text{and} \\ (a0'c) = a \vee c = c. \end{cases}$$

Now by (4.6), (5.3.10), (5.3.1) (5.3.3) (5.3.4), (4.9), (5.3.2), (5.3.10), we have

$$\begin{aligned} (5.3.11) \qquad a \vee (b \wedge c) &= (a0'(b0c)) \\ &= ((a0c)0'(b0c)) \\ &= (a(c0b)(0'c0)) \\ &= (a(c0b)c) \\ &= ((a0c)cb) \\ &= (acb). \end{aligned}$$

The preceding argument remains valid if a and c, $\wedge$ and $\vee$, 0 and $0'$ are interchanged.

Hence if $a \leq c$,

$$(5.3.12) \qquad c \wedge (b \vee a) = (cab).$$

By (5.3.3), (5.3.5), (5.3.11) and (5.3.12),

$$a \vee (b \wedge c) = (a \vee b) \wedge c,$$

and $(A,\wedge,\vee)$ is a modular lattice.

Also by (4.6), (4.2), (4.6), (5.3.3) (5.3.4) (4.9),

$$a \wedge 0 = (a00) = 0$$

$$a \wedge 0' = (a00') = a$$

which imply that the smallest and the greatest elements of A exist and equal to 0 and $0'$ respectively.

Besides, by (4.6), (5.3.3) (5.3.4) (4.9),

$$a \wedge a' = (a0a') = 0$$

$$a \vee a' = (a0'a') = 0'.$$

This shows that a' is a complement of a with respect to $(A,\wedge,\vee)$. Moreover, this is the only complement of a. For if a'' is a complement of a, then

$$(5.3.13) \qquad \begin{cases} a \wedge a'' = (a0a'') = 0 \\ \\ a \vee a'' = (a0'a'') = 0'. \end{cases}$$

Now by (4.9), (5.3.13), (5.3.1) (5.3.3)(5.3.4), (4.9) (5.3.13), (4.9),

$$a' = (0'a'0) = ((a0'a'')a'(a0a''))$$

$$= (0'(a'a''a)(a0a''))$$

$$= (0'a''0) = a''.$$

Thus $(A,\wedge,\vee)$ is a uniquely complemented modular lattice. But it is known that a uniquely complemented modular lattice is distributive. See, e.g., [13], p. 113, Theorem 51. Therefore $(A,\wedge,\vee)$ is a complemented distributive lattice.

Now it follows that

$$(a\wedge b) \vee (b\wedge c) \vee (c\wedge a) = (b\wedge c) \vee (a\wedge (b\vee c)).$$

As $b \wedge c \leq b \vee c$,

(5.3.14) $$(b\wedge c) \vee (a\wedge (b\vee c))$$

$$= ((b\wedge c)(b\vee c)a)$$

$$= ((b0c)(b0'c)a)$$

$$= ((abc)0(b0'c))$$

$$= (abc) \wedge (b0'c)$$

by (5.3.11), (4.6), (5.3.1) (5.3.3) (5.3.4), (4.6). But

$$(abc) \vee (b0'c) = ((abc)0'(b0'c))$$

$$= (a(b0'c)(b0'c))$$

$$= (b0'c)$$

by (4.6), (5.3.1) (5.3.3) (5.3.4), (4.2). Hence

$$(abc) \wedge (b0'c) = (abc).$$

Therefore by (5.3.14),

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (abc).$$

And by the results of page five A is a ternary Boolean algebra.

Conversely, if A is a ternary Boolean algebra then (4.1), (4.2), (4.4), (5.3.3) and (5.3.4) hold, hence (4.9) and (5.3.1) hold.

This completes the proof of Theorem 5.3.

Theorem 5.3 shows that the identities (4.2), (4.9) and (5.3.1) form a set of axioms for a Boolean algebra.

The following argument will show that these axioms are independent.

By A_1 (see p.9), (4.2) and (5.3.1) are valid but (4.9) is not satisfied.

A_4. Consider the set $B = \{u,v\}$, $u \neq v$ and define on it a ternary operation by $(abc) = u$ or v according as one or all

of a,b,c are u or v and a unary operation by $a' = a$ for all a belonging to B.

Then (4.9) and (5.3.1) are satisfied but (4.2) is false.

A_5. Let $B = \{0,1\}$ where 0 and 1 are Boolean zero and unit, respectively, and $0 \neq 1$, and let $(abc) = (a \vee c) \wedge b$, $0' = 1$ and $1' = 0$. Then (4.2) and (4.9) are satisfied, but (5.3.1) is not valid; for if $a = c = e = 1$ and $b = d = 0$,

$$((bde)(aed)c) = (((0 \vee 1) \wedge 0) \vee 1) \wedge ((1 \vee 0) \wedge 1) = 1,$$

whereas $(b(edc)(aed)) = (0 \vee ((1 \vee 0) \wedge 1)) \wedge ((1 \vee 1) \wedge 0) = 0.$

Many further sets of axioms of this type can be obtained. Three of them are stated below. The proof that these are sets of axioms for a Boolean algebra is omitted.

B_1.

$$(aba) = a$$

$$(abb') = a$$

$$((dbe)c(ead)) = ((dbe)a(cde))$$

B_2.

$$(aba) = a$$

$$(ab'b) = a$$

$$((edc)a(ebd)) = (c(ebd)(dae))$$

B_3. $$(aab) = a$$
$$(bb'a) = a$$
$$((dbe)c(ead)) = ((dbe)a(cde)).$$

In the following, it will only be shown that, in each of these three cases, the three axioms are independent.

A_6. Let $B = \{0,1\}$ where 0 and 1 are Boolean zero and unit $(0 \neq 1)$, and let the ternary operation be defined by $(abc) = (b \vee c) \wedge a$ and the unary operation by $0' = 1$ and $1' = 0$.

Then the postulates stated under B_1 as well as those stated under B_2 are independent by A_1, A_4 and A_6.

A_7. Consider the set $B = \{u,v,w\}$ of three different elements and define a ternary operation on B by $(abc) = a$ if $a = b$, $(abc) = c$ otherwise, and a unary operation on B by $v' = w' = u$, $u' = w$.

Then the axioms listed under B_3 are independent by A_1, A_4 and A_7. That the third axiom listed under B_3 is not valid in A_7 can be seen as follows: Let $a = b = c = w$; $d = u$ and $e = v$. Then $((dbe)c(ead)) = ((uwv)w(vwu)) = u$. On the other hand, $((dbe)a(cde)) = ((uwv)w(wuv)) = (vwv) = v$.

It should be observed that from each set of postulates for a ternary Boolean algebra mentioned in this section another

such set can be obtained by replacing every $(x_1x_2x_3)$ by $(x_{p(1)}x_{p(2)}x_{p(3)})$ where p is some permutation of $\{1,2,3\}$. The same applies to the sets of postulates for a Boolean algebra with which we shall deal in the next section.

6. SETS OF TWO POSTULATES

In this section sets of two independent postulates for a Boolean algebra in terms of one ternary operation and one unary operation are obtained. This is done by reducing each axiom set occuring in the preceding section to a set of two independent postulates in many different ways.

In the postulate set occuring in Theorem 5.1, (4.2) can be dispensed with by a suitable re-arrangement of (4.1), and a similar thing applies to the postulate set occuring in Theorem 5.2. Thus the five axioms given by Grau can be reduced to two postulates.

THEOREM 6.1. Let $\mathcal{A}$ be any algebraic system consisting of a set A, a ternary operation (abc) on A and a unary operation a' on A. Then $\mathcal{A}$ is a Boolean algebra if and only if

(4.4) $$(abb') = a$$

and one of

(4.1.3) $$(de(abc)) = (b(dea)(edc))$$

(4.1.4) $$(de(abc)) = (b(edc)(dea))$$

identically hold.

Proof. If A is a Boolean algebra (4.4), (4.1.3) and (4.1.4) obviously hold since (abc) is completely commutative.

Conversely, let us consider the case that (4.4) and one of (4.1.3) and (4.1.4) identically hold. If we show that, in either case, (baa) = a identically and the operation (abc) is completely commutative, then it will follow from Theorem 5.1 that A is a Boolean algebra.

First assume (4.4) and (4.1.3) . Then, by (4.4), (4.4), (4.1.3), (4.4),

(6.1.1) $$\begin{aligned} a = (abb') &= (ab(b'aa')) \\ &= (a(abb')(baa')) \\ &= (aab). \end{aligned}$$

By (6.1.1), (4.1.3), (6.1.1),

(6.1.2) $$\begin{aligned} (baa) &= (b(aaa)(aaa)) \\ &= (aa(aba)) = a. \end{aligned}$$

By (6.1.2), (4.1.3), (4.4) (6.1.2), (6.1.1),

(6.1.3) $$\begin{aligned}(aba) &= (ab(b'aa))\\ &= (a(abb')(baa))\\ &= (aaa) = a.\end{aligned}$$

By (6.1.3), (4.1.3), (6.1.3),

(6.1.4) $$\begin{aligned}(abc) &= (ab(cyc))\\ &= (y(abc)(bac)) = (bac)\end{aligned}$$

if $y = (bac)$.

By (4.4) and (6.1.4),

(6.1.5) $$(bab') = a.$$

By (6.1.5), (4.1.3), (6.1.2) (6.1.5),

(6.1.6) $$\begin{aligned}(abc) &= (ab(bcb'))\\ &= (c(abb)(bab'))\\ &= (cba).\end{aligned}$$

By (6.1.4) and (6.1.6), (abc) is completely commutative. This completes the proof of the first half of the theorem.

Now assume (4.4) and (4.1.4). By (4.4), (4.4), (4.1.4), (4.4),

(6.1.7)
$$\begin{aligned} a = (abb') &= (ab(b'aa')) \\ &= (a(baa')(abb')) \\ &= (aba). \end{aligned}$$

By (4.4), (4.1.4), (4.4), (4.1.4), (6.1.7), (6.1.7),

(6.1.8)
$$\begin{aligned} (aab) &= (aa(baa')) \\ &= (a(aaa')(aab)) \\ &= (aa(aab)) \\ &= (a(aab)(aaa)) \\ &= (a(aab)a) = a. \end{aligned}$$

By (6.1.7), (4.1.4), (6.1.8),

(6.1.9)
$$\begin{aligned} (abc) &= (ab(cyc)) \\ &= (y(bac)(abc)) \\ &= (bac) \end{aligned}$$

if $y = (bac)$.

By (6.1.7) and (6.1.9),

(4.2)
$$(baa) = a.$$

By (6.1.8), (4.1.4), (4.2), (4.1.4), (4.2),

$$(6.1.10) \qquad \begin{aligned}(abc) &= (ab(cca)) \\ &= (c(baa)(abc)) \\ &= (ca(abc)) \\ &= (b(acc)(caa)) \\ &= (bca).\end{aligned}$$

By (6.1.9) and (6.1.10), (abc) is completely commutative.

Thus in both cases (4.2) holds and (abc) is invariant under all permutations of a, b and c. Hence the proof is complete.

Theorem 6.1 shows that the axioms listed in Theorem 5.1 or 5.2 can be reduced to two identities namely (4.4) and a rearrangement of (4.1).

Either set of axioms involved in Theorem 6.1 is independent by A_1 and A_6.

In the following we shall see that the axioms listed in Theorem 5.1 can also be reduced to (4.2) and a combination of (4.1) and (4.4).

THEOREM 6.2. The axioms

$$(4.2) \qquad (baa) = a$$

and one of the following identities,

(6.2.1) $$(de(abc)) = (((dea)b(dec))xx')$$

(6.2.2) $$(de(abc)) = ((dea)(bxx')((dec)xx'))$$

(6.2.3) $$(de(abc)) = (((dea)xx')(bxx')((dec)xx'))$$

(6.2.4) $$((de(abc))xx') = ((dea)b(dec))$$

(6.2.5) $$(d(exx')((abc)xx')) = ((dea)b(dec))$$

(6.2.6) $$((dxx')(exx')((abc)xx')) = ((dea)b(dec))$$

define a Boolean algebra.

Proof. We shall prove this theorem by showing that, in each case, the axiom set is equivalent to the hypotheses of Theorem 5.1. For this purpose, it will be sufficient to show that (4.1) and (4.4) hold in each case.

Let (*) be any one of (6.2.1) to (6.2.6). Assume (4.2) and (*). Put $b = c = d = e = a$ in (*) and use (4.2). Then we obtain (4.4) and this together with (*) implies (4.1) and the proof of the theorem is complete.

Each of the six sets of postulates involved in Theorem 6.2 is independent by A_2 and A_5.

In the next two theorems, twelve sets of axioms for the ternary operation (abc) and the unary operation a' are stated and it is shown that each of these axiom sets is equivalent to the set of the axioms listed in Theorem 5.2. Therefore each of these axiom sets is a set of postualtes for a Boolean algebra.

THEOREM 6.3. The set of two identities consisting of

(4.2) $$(baa) = a$$

and one of

(6.3.1) $$(de(abc)) = ((a(deb)(dec))xx')$$

(6.3.2) $$(de(abc)) = (a((deb)xx')((dec)xx'))$$

(6.3.3) $$(de(abc)) = ((axx')((deb)xx')((dec)xx'))$$

(6.3.4) $$((de(abc))xx') = (a(deb)(dec))$$

(6.3.5) $$(d(exx')((abc)xx')) = (a(deb)(dec))$$

(6.3.6) $$((dxx')(exx')((abc)xx')) = (a(deb)(dec))$$

is equivalent to the set of the following three identities:

(4.2) $(baa) = a$

(4.4) $(abb') = a$

(4.1.2) $(de(abc)) = (a(deb)(dec))$.

Proof. The proof is similar to the proof of Theorem 6.2. Each of the six sets of axioms involved in Theorem 6.3 is independent by A_2 and A_5.

THEOREM 6.4. The set of the axioms

(6.4.1) $((bxx')aa) = a$

and one of the following identities

(6.4.2) $(de(abc)) = ((a(deb)(dec))xx')$

(6.4.3) $(de(abc) = (a((deb)xx')((dec)xx'))$

(6.4.4) $((de(abc)) = ((axx')((deb)xx')((dec)xx'))$

(6.4.5) $((de(abc))xx') = (a(deb)(dec))$

(6.4.6) $(d(exx')((abc)xx')) = (a(deb)(dec))$

(6.4.7) $$((dxx')(exx')((abc)xx')) = (a(deb)(dec))$$

is equivalent to the set of the axioms listed in Theorem 5.2.

Proof. It is sufficient to show that (4.2), (4.4) and (4.1.2) hold in each case.

Let (*) be any one of (6.4.2) to (6.4.7). Assume (6.4.1) and (*). Put $a = d = (yxx')$ and $b = c = e$ in (*) and use (6.4.1). Then we obtain (4.4), and this together with (6.4.1) and (*) implies (4.2) and (4.1.2).

Each of the six sets of axioms involved in Theorem 6.4 is independent by A_2 and A_5.

The axioms listed in Theorem 5.3 and other sets of axioms of this type can be transformed into sets of two postulates by combining axioms of the types (4.9) and (5.3.1). For the axioms listed in Theorem 5.3, this is illustrated by the following theorem:

THEOREM 6.5. The set of two identities consisting of

(4.2) $$(baa) = a$$

and one of the following identities

(6.5.1) $$((bde)(aed)c) = (x'(b(edc)(aed))x)$$

(6.5.2) $((bde)(aed)c) = (b(x'(edc)x)(x'(aed)x))$

(6.5.3) $((bde)(aed)c) = ((x'bx)(x'(edc)x)(x'(aed)x))$

(6.5.4) $(x'((bde)(aed)c)x) = (b(edc)(aed))$

(6.5.5) $((bde)(x'(aed)x)(x'cx)) = (b(edc)(aed))$

(6.5.6) $((x'(bde)x)(x'(aed)x)(x'cx)) = (b(edc)(aed))$

is equivalent to the set of axioms listed in Theorem 5.3.

Proof. The proof is similar to the proof of Theorem 6.2.

Each of these six sets of identities is independent by A_4 and A_5.

CHAPTER III

SETS OF AXIOMS FOR DISTRIBUTIVE LATTICES

7. INTRODUCTION

According to G. Birkhoff and S.A. Kiss [2] if 0 and
if
I are elements of A, (abc) is a ternary operation on A which satisfies the identities

(7.1) $$(0aI) = a$$

(7.2) $$(aba) = a$$

(7.3) $$(abc) = (bca)$$

(7.4) $$(abc) = (bac)$$

(7.5) $$((abc)de) = ((ade)b(cde))$$

if
and $\wedge$ and $\vee$ are the binary operations on A defined by

(7.6) $$\begin{cases} a \wedge b = (a0b) \\ a \vee b = (aIb) \end{cases}$$

then $(A,\wedge,\vee)$ is a distributive lattice, the lower and upper bounds of A with respect to the corresponding lattice ordering of A exist and are 0 and I, respectively, and

$$(7.7) \quad (abc) = (a\wedge b) \vee (b\wedge c) \vee (c\wedge a) = (a\vee b) \wedge (b\vee c) \wedge (c\vee a)$$

identically. Conversely, if $(A,\wedge,\vee)$ is a distributive lattice, if the lower and upper bounds of A with respect to the corresponding lattice ordering of A exist and are denoted. 0 and I, respectively, and if (abc) is the ternary operation on A defined by (7.7), then (7.1) to (7.6) hold identically.

Later on in [1] , Birkhoff posed the following problem as 64: "Show that at least one of (7.3) and (7.4) can be dispensed with if, a suitable permutation of (7.5) is used". This question has been answered by many authors and the following axiom systems solving Birkhoff's problem have been given.

(7.1), (7.2) and one of the following:

ph. Vassiliou [14]: $(d(abc)e) = ((edc)b(eda))$

R. Croisot [5] : $(d(abc)e) = (b(cde)(ade))$

J. Hashimoto [7] : $(d(abc)e) = ((ebd)a(ecd))$

B. Sobocinski [12]: $(d(abc)e) = ((dce)(dae)b)$.

In all these systems of postulates both (7.3) and (7.4) are eliminated by a suitable rearrangement of (7.5).

M. Sholander [10] and B. Sobocinski [12] have also stated some sets of two postulates in five variables in terms of a ternary operation (abc) which are equivalent to (7.1) to (7.5). Sholander stated without proof that

$$(Oa(IbI)) = a$$
$$(d(abc)e) = ((dbe)c(ade))$$

is such a postulate set.

Sobocinski has proved that each of the following five sets of identities has the property stated above.

(i) $$(O(baa)I) = a$$
$$(d(abc)e) = ((dce)(dae)b)$$

(ii) $$(O(aab)I) = a$$
$$(d(abc)e) = ((dce)(dae)b)$$

(iii) $$(O(aba)I) = a$$
$$(d(abc)e) = ((dce)(dae)b)$$

(iv) $$(aba) = a$$
$$(O(d(abc)e)I) = ((dce)(dae)b)$$

(v) $(aba) = a$

$$(d(abc)e) = (O((dce)(dae)b)I).$$

In this thesis a number of sets of two to five postulates for a distributive lattice with O and I are stated and added to this list. A few of them are shown to be such postulate sets, while the rest of them are only listed. Most of them do not involve one or both of (7.3) and (7.4) and consequently, they are solutions to Birkhoff's problem.

8. SOLUTIONS OF BIRKHOFF'S PROBLEM 64

In this section only axiom sets which are solutions to Birkhoff's problem are given.

Consider re-arrangements of (7.5) of the following sort:

(i) The left side $((abc)de)$ or $(d(abc)e)$ or $(de(abc))$.

(ii) The right side is of the form

$((x_1x_2x_3)x_4(x_5x_6x_7))$ or of the form

$((x_1x_2x_3)(x_5x_6x_7)x_4)$ or of the form

$(x_4(x_1x_2x_3)(x_5x_6x_7))$

where $(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$ is some re-arrangement of (a,d,e,b,c,d,e)

(iii) d and e occur as factors in $(x_1x_2x_3)$ as well as in $(x_5x_6x_7)$.

(iv) d and e occur in $(x_5x_6x_7)$ in the same positions as in $(x_1x_2x_3)$.

According to this formulation, there are three possible left sides. On the right side, a, b and c can occur in six different orders. At each such order there are six possible arrangements of the factors of $(x_1x_2x_3)$, each of which corresponds to exactly one arrangement of the factors of $(x_5x_6x_7)$. Hence there are $6 \times 6 \times 3$ possible right sides, therefore, $6 \times 6 \times 3 \times 3 = 324$ re-arrangements of (7.5) of the sort described.

If the order of a, b, c on the right side of an identity replacing (7.5) is different from the alphabetic order, then let us say that a, b, c are permuted in this identity. Similarly, if the order of a, d, e, on the left side of an equation replacing (7.5) is different from their order in $(x_1x_2x_3)$ if a occurs in this product and different from their order in the product arising from $(x_1x_2x_3)$ by replacing b or c by a if b or c occurs in $(x_1x_2x_3)$, then let us say that a, d, e are permuted in this equation. If the permutation of (a,b,c) or (a,d,e) involved is a cyclic one, let us say that a, b, c or a, d, e, respectively, are cyclically permuted in the equation considered.

Further, let us say that a permutation of (a,b,c) and

a permutation of (a,d,e) are of the same type if the second arises from the first by replacing b and c by d and e, respectively. For example, the permutations (b,c,a) and (d,e,a) of (a,b,c) and (a,d,e), respectively, are of the same type. Otherwise, let us say that the permutations are of different types.

Now all those 324 identities which are rearrangements of (7.5) can be divided into the following four groups:

G_1. Those identities in which a, b, c or a, d, e are permuted, and a, b, c as well as a, d, e are cyclically permuted if they are permuted.

G_2. The identities in which both a, b, c and a, d, e are permuted, the two permutations are of different types, and atleast one of them is not cyclic.

G_3. The identities in which a, b, c or a, d, e are permuted, a, b, c as well as a, d, e are not cyclically permuted if they are permuted, and if both a, b, c and a, d, e are permuted, the two permutations involved are of the same type.

G_4. The identities in which neither a, b, c nor a, d, e are permuted.

Again among those 324 re-arrangements of (7.5), consider identities of the following five types:

(i) the identities in which (abc) on the left side and the single factor in the ternary product on the right side are in the same positions but not in the central positions of the ternary products, the single factor in the ternary product on the right side is different from the factor in the same position in (abc), and d and e are in the same positions in the three ternary products in which they occur as factors;

(ii) the identities of type (i) with the positions of d and e interchanged on the right side;

(iii) the identities in which (abc) and the single factor in the ternary product on the right are in the same positions but not in the centres of the ternary products and the positions of d and e interchanged on one side from their positions in the ternary product or products on the other side;

(iv) the identities in which the single factor in the ternary product on the right side is either a or c and is not in the central position of this product, and in which the permutation of (a,b,c) or that of (a,d,e) is

of type (a,c,b)	of type (c,b,a) or of type (a,c,b)	of type (b,a,c)	of type (c,b,a) or of type (b,a,c)

if the single factor in the ternary product on the right side is

a	c

and (abc) is

in the central position	not in the central position	in the central position	not in the central position

of the ternary product on the left side;

(v) the identities in which the single factor in the right ternary product is b and is in the centre but (abc) is not in the middle of the left ternary product and the permutation of (a,b,c) or that of (a,d,e) is of the type (c,b,a).

The author of this thesis has proved that each of the following identities together with (7.1) and (7.2) forms a set of three independent postulates for a distributive lattice with O and I, the lattice operations defined by (7.6):

(I) all identities which belong to group G_1 and are not of type (i);

(II) all identities which belong to group G_2 and are not of type (ii);

(III) all identities which belong to group G_3 and are of type (iv) or (v).

The author has also proved that each of the following identities together with (7.1) to (7.3) forms a set of four independent postulates for a distributive lattice with 0 and I, the lattice operations defined in the same way as before.

(IV) all identities which belong to group G_2 and are of type (ii) and also all identities which belong to group G_3 and are not of type (iii), (iv) or (v).

The identities mentioned under (I) are the following:

(1) $((abc)de) = (a(deb)(dec))$

(2) $= (a(ebd)(ecd)$

(3) $= ((ecd)a(ebd))$

(4) $= ((dec)a(deb))$

(5) $= ((cde)a(bde))$

(6) $= ((ebd)(ecd)a)$

(7) $= ((deb)(dec)a)$

(8) $= ((bde)(cde)a)$

(9) $= ((ead)b(ecd))$

(10) $= ((dea)b(dec))$

(11) $= ((ecd)(ead)b)$

(12) $= ((dec)(dea)b)$

(13) $= ((cde)(ade)b)$

(14) $((abc)de) = (b(ecd)(ead))$

(15) $= (b(dec)(dea))$

(16) $= ((ead)(ebd)c)$

(17) $= ((dea)(deb)c)$

(18) $= (c(ead)(ebd))$

(19) $= (c(dea)(deb))$

(20) $= ((ebd)c(ead))$

(21) $= ((deb)c(dea))$

(22) $= ((bde)c(ade))$

(23) $(de(abc)) = (a(ebd)(ecd))$

(24) $= (a(bde)(cde))$

(25) $= ((ecd)a(ebd))$

(26) $= ((dec)a(deb))$

(27) $= ((cde)a(bde))$

(28) $= ((ebd)(ecd)a)$

(29) $= ((bde)(cde)a)$

(30) $= ((ead)b(ecd))$

(31) $= ((ade)b(cde))$

(32) $= ((ecd)(ead)b)$

(33) $= ((cde)(ade)b)$

(34) $= (b(ecd)(ead))$

(35) $= (b(dec)(dea))$

(36) $= (b(cde)(ade))$

(37) $= ((ead)(ebd)c)$

(38) $= ((ade)(bde)c)$

(39) $(de(abc)) = (c(ead)(ebd))$

(40) $= (c(dea)(deb))$

(41) $= (c(ade)(bde))$

(42) $= ((ebd)c(ead))$

(43) $= ((deb)c(dea))$

(44) $= ((bde)c(ade))$

(45) $(d(abc)e) = (a(edb)(edc))$

(46) $= (a(bed)(ced))$

(47) $= ((dce)a(dbe))$

(48) $= ((edc)a(edb))$

(49) $= ((ced)a(bed))$

(50) $= ((dbe)(dce)a)$

(51) $= ((edb)(edc)a)$

(52) $= ((bed)(ced)a)$

(53) $= ((eda)b(edc))$

(54) $= ((aed)b(ced))$

(55) $= ((dce)(dae)b)$

(56) $= ((edc)(eda)b)$

(57) $= ((ced)(aed)b)$

(58) $= (b(dce)(dae))$

(59) $= (b(edc)(eda))$

(60) $= (b(ced)(aed))$

(61) $= ((eda)(edb)c)$

(62) $= ((aed)(bed)c)$

(63) $= (c(dae)(dbe))$

(64) $(d(abc)e) = (c(eda)(edb))$

(65) $= (c(aed)(bed))$

(66) $= ((dbe)c(dae))$

(67) $= ((edb)c(eda))$

(68) $= ((bed)c(aed)).$

The identities mentioned under (II) are the following:

(69) $((abc)de) = (a(dec)(deb))$

(70) $= (a(edc)(edb))$

(71) $= (a(dce)(dbe))$

(72) $= (a(ecd)(ebd))$

(73) $= ((dce)a(dbe))$

(74) $= ((ebd)a(ecd))$

(75) $= ((deb)a(dec))$

(76) $= ((edb)a(edc))$

(77) $= ((edc)a(edb))$

(78) $= ((bed)a(ced))$

(79) $= ((ced)a(bed))$

(80) $= ((dbe)(dce)a)$

(81) $= ((dce)(dbe)a)$

(82) $= ((ecd)(ebd)a)$

(83) $= ((dec)(deb)a)$

(84) $= ((edb)(edc)a)$

(85) $= ((bed)(ced)a)$

(86) $((abc)de) = ((ced)(bed)a)$

(87) $= ((ecd)b(ead))$

(88) $= ((dce)b(dae))$

(89) $= ((dec)b(dea))$

(90) $= ((ced)b(aed))$

(91) $= ((dce)(dae)b)$

(92) $= ((ead)(ecd)b)$

(93) $= ((dae)(dce)b)$

(94) $= ((edc)(eda)b)$

(95) $= ((dea)(dec)b)$

(96) $= ((eda)(edc)b)$

(97) $= ((ced)(aed)b)$

(98) $= (b(dce)(dae))$

(99) $= (b(ead)(ecd))$

(100) $= (b(edc)(eda))$

(101) $= (b(dea)(dec))$

(102) $= (b(eda)(edc))$

(103) $= ((ebd)(ead)c)$

(104) $= ((deb)(dea)c)$

(105) $= ((edb)(eda)c)$

(106) $= ((bed)(aed)c)$

(107) $= (c(dae)(dbe))$

(108) $= (c(ebd)(ead))$

(109) $= (c(dbe)(dae))$

(110) $= (c(eda)(edb))$

(111) ((abc)de) = (c(deb)(dea))

(112) = ((dbe)c(dae))

(113) = ((dae)c(dbe))

(114) = ((ead)c(ebd))

(115) = ((edb)c(eda))

(116) = ((dea)c(deb))

(117) = ((eda)c(edb))

(118) = ((bed)c(aed))

(119) (de(abc)) = (a(edc)(edb))

(120) = (a(ecd)(ebd))

(121) = (a(cde)(bde))

(122) = (a(ced)(bed))

(123) = ((dbe)a(dce))

(124) = ((dce)a(dbe))

(125) = ((ebd)a(ecd))

(126) = ((edc)a(edb))

(127) = ((bde)a(cde))

(128) = ((bed)a(ced))

(129) = ((ced)a(bed))

(130) = ((dce)(dbe)a)

(131) = ((ecd)(ebd)a)

(132) = ((dbe)(dce)a)

(133) = ((bed)(ced)a)

(134) = ((cde)(bde)a)

(135) = ((ecd)b(ead))

(136) = ((dce)b(dae))

(137) $(de(abc)) = ((edc)b(eda))$

(138) $= ((cde)b(ade))$

(139) $= ((dce)(dae)b)$

(140) $= ((ead)(ecd)b)$

(141) $= ((ced)(aed)b)$

(142) $= ((aed)(ced)b)$

(143) $= ((ade)(cde)b)$

(144) $= (b(dce)(dae))$

(145) $= (b(dae)(dce))$

(146) $= (b(ead)(ecd))$

(147) $= (b(edc)(eda))$

(148) $= (b(ced)(aed))$

(149) $= (b(aed)(ced))$

(150) $= (b(ade)(cde))$

(151) $= ((ebd)(ead)c)$

(152) $= ((dbe)(dae)c)$

(153) $= ((bed)(aed)c)$

(154) $= ((bde)(ade)c)$

(155) $= (c(dae)(dbe))$

(156) $= (c(dbe)(dae))$

(157) $= (c(ebd)(ead))$

(158) $= (c(eda)(edb))$

(159) $= (c(edb)(eda))$

(160) $= (c(aed)(bed))$

(161) $= (c(bde)(ade))$

(162) $(de(abc)) = ((dbe)c(dae))$

(163) $= ((ead)c(ebd))$

(164) $= ((edb)c(eda))$

(165) $= ((eda)c(edb))$

(166) $= ((bed)c(aed))$

(167) $= ((aed)c(bed))$

(168) $= ((ade)c(bde))$

(169) $(d(abc)e) = (a(edc)(edb))$

(170) $= (a(ecd)(ebd))$

(171) $= a(ced)(bed))$

(172) $= (a(cde)(bde))$

(173) $= ((ebd)a(ecd))$

(174) $= ((ecd)a(ebd))$

(175) $= ((deb)a(dec))$

(176) $= ((dec)a(deb))$

(177) $= ((edb)a(edc))$

(178) $= ((bed)a(ced))$

(179) $= ((cde)a(bde))$

(180) $= ((ebd)(ecd)a)$

(181) $= ((deb)(dec)a)$

(182) $= ((dec)(deb)a)$

(183) $= ((edc)(edb)a)$

(184) $= ((ced)(bed)a)$

(185) $= ((bde)(cde)a)$

(186) $= ((cde)(bde)a)$

(187)	(d(abc)e)	= ((edc)b(eda))
(188)		= ((dec)b(dea))
(189)		= ((ced)b(aed))
(190)		= ((cde)b(ade))
(191)		= ((ecd)(ead)b)
(192)		= ((ead)(ecd)b)
(193)		= ((dec)(dea)b)
(194)		= ((eda)(edc)b)
(195)		= ((cde)(ade)b)
(196)		= ((aed)(ced)b)
(197)		= ((ade)(cde)b)
(198)		= (b(ecd)(ead))
(199)		= (b(ead)(ecd))
(200)		= (b(dec)(dea))
(201)		= (b(dea)(dec))
(202)		= (b(eda)(edc))
(203)		= (b(cde)(ade))
(204)		= (b(aed)(ced))
(205)		= ((ebd)(ead)c)
(206)		= ((edb)(eda)c)
(207)		= ((deb)(dea)c)
(208)		= ((bed)(aed)c)
(209)		= (c(ead)(ebd))
(210)		= (c(dea)(deb))
(211)		= (c(deb)(dea))

(212)	$(d(abc)e) = (c(edb)(eda))$
(213)	$= (c(ade)(bde))$
(214)	$= ((eda)c(edb))$
(215)	$= ((bde)c(ade))$
(216)	$= ((aed)c(bed))$
(217)	$= ((ade)c(bde))$
(218)	$= (c(bde)(ade))$
(219)	$= (c(bed)(aed))$
(220)	$= ((ebd)c(ead))$
(221)	$= ((ead)c(ebd))$
(222)	$= ((deb)c(dea))$

The identities named under (III) are the following:

(223)	$((abc)de) = (a(edb)(edc))$
(224)	$= ((edc)(edb)a)$
(225)	$= ((cde)(bde)a)$
(226)	$= ((eda)b(edc))$
(227)	$= ((edc)b(eda))$
(228)	$= ((cde)b(ade))$
(229)	$= ((dae)(dbe)c)$
(230)	$= ((dbe)(dae)c)$
(231)	$= ((eda)(edb)c)$
(232)	$= ((bde)(ade)c)$
(233)	$= (c(edb)(eda))$

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(234) $(de(abc)) = (a(dbe)(dce))$

(235) $= (a(dec)(deb))$

(236) $= (a(dce)(dbe))$

(237) $= (a(bed)(ced))$

(238) $= ((ced)(bed)a)$

(239) $= ((dec)b(dea))$

(240) $= ((aed)b(ced))$

(241) $= ((ced)b(aed))$

(242) $= ((aed)(bed)c)$

(243) $= (c(deb)(dea))$

(244) $= (c(bed)(aed))$

(245) $(d(abc)e) = (a(deb)(dec))$

(246) $= (a(dec)(deb))$

(247) $= (a(dce)(dbe))$

(248) $= ((dbe)(dae)c)$

(249) $= ((ade)(bde)c)$

(250) $= ((bde)(ade)c)$

The identities named under (IV) are the following:

(251) $((abc)de) = (a(dbe)(dce))$

(252) $= ((dbe)a(dce))$

(253) $= ((bde)a(cde))$

(254) $= ((dae)b(dce))$

(255) $= ((aed)b(ced))$

(256) $((abc)de) = ((aed)(ced)b)$

(257) $= ((ade)(cde)b)$

(258) $= (b(dae)(dce))$

(259) $= (b(ced)(aed))$

(260) $= (b(aed)(ced))$

(261) $= ((aed)(bed)c)$

(262) $= (c(aed)(bed))$

(263) $= (c(bed)(aed))$

(264) $= ((aed)c(bed))$

(265) $= ((ade)c(bde))$

(266) $(de(abc)) = (a(edb)(edc))$

(267) $= ((edb)a(edc))$

(268) $= ((deb)a(dec))$

(269) $= ((edb)(edc)a)$

(270) $= ((edc)(edb)a)$

(271) $= ((dae)b(dce))$

(272) $= ((eda)b(edc))$

(273) $= ((dae)(dce)b)$

(274) $= ((edc)(eda)b)$

(275) $= ((eda)(edc)b)$

(276) $= (b(dea)(dec))$

(277) $= (b(eda)(edc))$

(278) $= ((dae)(dbe)c)$

(279) $= ((dae)c(dbe))$

(280) $= ((dea)c(deb))$

(281) $(d(abc)e) = (a(ebd)(ecd))$

(282) $= (a(bde)(cde))$

(283) $= ((dbe)a(dce))$

(284) $= ((bde)a(cde))$

(285) $= ((dce)(dbe)a)$

(286) $= ((ecd)(ebd)a)$

(287) $= ((ead)b(ecd))$

(288) $= ((ecd)b(ead))$

(289) $= ((dce)b(dae))$

(290) $= ((dea)b(dec))$

(291) $= ((ade)b(cde))$

(292) $= ((dae)(dce)b)$

(293) $= ((dea)(dec)b)$

(294) $= (b(dae)(dce))$

(295) $= (b(ade)(cde))$

(296) $= ((ead)(ebd)c)$

(297) $= ((dea)(deb)c)$

(298) $= (c(ebd)(ead))$

(299) $= (c(dbe)(dae))$

(300) $= ((dae)c(dbe))$

(301) $= ((dea)c(deb))$

Because of lack of space the proof that each of these 301 postulate sets does define a distributive lattice with 0 and I, the lattice operations defined by (7.6), cannot be

produced in this thesis. In the following the proof will be given for the postulate sets associated with (110), (226), (237), (253), and (278).

THEOREM 8.1. Let $\mathcal{A}$ be an algebraic system consisting of a set A with elements 0 and I and a ternary operation (abc) satisfying the identities,

(7.1) $$(0aI) = a$$

(7.2) $$(aba) = a$$

and one of

(226) $$((abc)de) = ((eda)b(edc))$$

(110) $$((abc)de) = (c(eda)(edb))$$

(237) $$(de(abc)) = (a(bed)(ced)).$$

Let $\wedge$ and $\vee$ be the binary operations on A defined by (7.6). Then $(A,\wedge,\vee)$ is a distributive lattice, and 0 and I have the properties stated at the beginning of section 7.

Proof. It is sufficient to prove (7.3) to (7.5). Assume (7.1), (7.2) and (226). Replace b and c by a in (226) and use (7.2) to get

(8.1.1) $$(ade) = (eda).$$

Then (7.5) follows by (226) and (8.1.1). By (7.1), (226), (7.1) (8.1.1) (7.2), (7.1),

(8.1.2) $$\begin{aligned}(a0I) &= ((0aI)0I)\\ &= ((I00)a(I0I))\\ &= (0aI) = a.\end{aligned}$$

By (7.1), (226) (7.2) (7.1), (7.2),

(8.1.3) $$\begin{aligned}(a00) &= ((0aI)00)\\ &= (0a0) = 0.\end{aligned}$$

By (8.1.2) (8.1.3), (226), (7.1) (8.1.1) (7.2),

(8.1.4) $$\begin{aligned}(aa0) &= ((a0I)a(a00))\\ &= ((Ia0)0a)\\ &= a.\end{aligned}$$

By (7.1), (226), (7.1) (8.1.1) (7.2) (7.2),

(8.1.5) $$\begin{aligned}(aII) &= ((0aI)II)\\ &= ((II0)a(III))\\ &= I.\end{aligned}$$

By (7.1), (226), (7.2) (7.1) (7.1),

(8.1.6) $$(aI0) = ((0aI)I0) = ((0I0)a(0II)) = a.$$

By (8.1.6) (8.1.5), (226), (7.1) (7.2),

(8.1.7) $$(aaI) = ((aI0)a(aII)) = ((0aI)Ia) = a.$$

Then by (8.1.1), (7.1), (226), (8.1.4) (8.1.7) (7.2),

(8.1.8) $$(aab) = (baa) = ((0bI)aa) = ((aa0)b(aaI)) = a.$$

Now by (8.1.1), (8.1.8), (226) (8.1.1) (8.1.8), (226) (8.1.1) (8.1.8), (8.1.8), (226) (8.1.8),

(8.1.9) $$(abc) = (cba) = ((ccb)ba) = ((cba)cb) = (cb(acb)) = ((acc)b(acb)) = (bca).$$

Now (8.1.9) includes (7.3), and (8.1.1) and (8.1.9) imply

(7.4).

Next assume (7.1), (7.2) and (110). By (7.2) and (110),

$$
\begin{aligned}
(8.2.7) \qquad (baa) &= (b(aaa)(aaa)) \\
&= ((aab)aa) \\
&= (((aaa)ab)aa) \\
&= ((a(baa)(baa))aa) \\
&= ((baa)a(aa(baa))) \\
&= (a((aa(baa))ab)((aa(baa))aa)) \\
&= (a((aa(baa))ab)((baa)aa)) \\
&= (a((aa(baa))ab)(a(aab)a)) \\
&= (a((aa(baa))ab)a) \\
&= a.
\end{aligned}
$$

By putting $a = b = c$ in (110) and using (7.2) and (8.2.7) we get

$$(8.2.8) \qquad (ade) = (eda).$$

By (8.2.7), (110) (8.2.7), (8.2.8) (110), (8.2.7),(8.2.8) (7.2), (110), (8.2.8) (7.1) (7.2),

(8.2.9) $$\begin{aligned}(a0I) &= ((0aa)0I)\\ &= (a0(I0a))\\ &= (a(a0I)(a00))\\ &= (a(a0I)0)\\ &= (0(a0I)(a0a))\\ &= ((Ia0)0a) = a.\end{aligned}$$

Now by (8.2.9), (110), (8.2.9) (8.2.8),

(8.2.10) $$\begin{aligned}(abc) &= ((abc)0I)\\ &= (c(I0a)(I0b))\\ &= (cab).\end{aligned}$$

(8.2.8), (8.2.10) and (110) imply (7.3) to (7.5).

Lastly assume (7.1), (7.2) and (237). By (7.1), (237), (7.1) (7.2),

(8.2.1) $$\begin{aligned}(I00) &= (I0(00I))\\ &= (0(00I)(I0I))\\ &= 0.\end{aligned}$$

By (7.1), (237), (8.2.1) (7.2),

(8.2.2) $$\begin{aligned}(00b) &= (00(0bI))\\ &= (0(b00)(I00))\\ &= 0.\end{aligned}$$

By (7.2), (237), (8.2.2) (7.2),

$$(8.2.3)\qquad (b00) = (b0(0b0)) = (0(b0b)(00b)) = 0.$$

By (7.1), (237), (8.2.3), (8.2.3), (237), (7.1) (7.2),

$$(8.2.4)\qquad (baa) = (b(0aI)(0aI)) = (Ia(b00)) = (Ia0) = (Ia(a00)) = (a(0aI)(0aI)) = a.$$

Now replace b and c in (237) by a and use (8.2.4) to get

$$(8.2.5)\qquad (dea) = (aed).$$

By (8.2.5), (7.2), (237) (7.2), (237) (8.2.5) (8.2.4), (8.2.5) (8.2.4), (237), (8.2.4) (8.2.5),

$$(8.2.6)\qquad (abc) = (cba) = (cb(aca)) = (ac(abc)) = (a(bca)c) = (a(acb)(ccb)) = (bc(aac)) = (bca).$$

Now (8.2.6) includes (7.3), (8.2.5) and (8.2.6) imply (7.4), and (7.5) follows from (237) , (8.2.5) and (8.2.6).

Let us now consider some independence examples.

A_8. Suppose $A = \{0,1\}$ where 0 and 1 are Boolean zero and unit $(0 \neq 1)$, and let 0 as well as I be the identity element of the Boolean join operation $\vee$ (the Boolean zero), and define (abc) as $a \vee b \vee c$. Then (7.2) does not hold, but (7.1) and the identities (1) to (250) are all satisfied.

A_9. Again assume a two element Boolean algebra A, and let 0 and I be the identity element of $\vee$ and $\wedge$, respectively (the Boolean zero and unit), and define (abc) as $(a \vee b) \wedge c$. Then (7.1) and (7.2) are valid, while the identities (1) to (250) all fail to hold. The non-validity of the identities (1) to (250) can be seen as follows: let the letters appearing in the ternary products on the left and right sides of the identity considered by x_1, x_2, x_3, x_4, x_5, and $y_1, y_2, y_3, y_4, y_5, y_6, y_7$, respectively, in the order in which they occur. There are two possibilities according as x_5 and y_7 are equal or unequal. Consider the case where $x_5 = y_7$. x_5 can only be c or e. If

$x_5 = c$	$x_5 = e$

replace all the letters occuring in the identity by the Boolean unit, except

x_3 and x_4 ,	y_5 and y_6 ,

which are replaced by the Boolean zero. Then it is easily seen that the two sides of the identity are unequal. If $x_5 \neq y_7$, let $y_7 = 0$, and let the other four of a,b,c,d,e be equal to I. Then the left side of the equation considered equals I, while the right side equals 0.

A_{10}. Consider the set $A = \{0,1\}$ where 0 and 1 are Boolean zero and unit respectively, $0 \neq 1$. Let 0 as well as I be either the Boolean zero or unit. Let (abc) be defined as 0 or 1 according as atleast two of a,b,c are 0 or 1. Then (7.1) does not hold whereas (7.2) and the identities (1) to (250) are all satisfied. The validity of the identities (1) to (250) can be proved as follows: as the operation (abc) is here completely commutative, each of (1) to (250) is equivalent to (7.5) and it is easily seen that (7.5) is satisfied.

By A_8 to A_{10}, the set of postulates for a distributive lattice with 0 and I consisting of (7.1), (7.2) and any one of (1) to (250) is independent.

Let us now consider sets of four postulates.

THEOREM 8.2. The set of the axioms (7.1) to (7.3) and one of

(253) $$((abc)de) = ((bde)a(cde))$$

(278) $$(de(abc)) = ((dae)(dbe)c)$$

is equivalent to the set of the axioms (7.1) to (7.5).

Proof. Assume (7.1) to (7.3) and (253). Then by (253) $d = I$, $e = 0$ and by (7.1) and (7.3) $(abc) = (bac)$, which is (7.4). (7.3), (7.4) and (253) imply (7.5).

Assume (7.1) to (7.3) and (278). Then $(aab) = (aba) = a$ by (7.3), (7.2). Putting $a = b = c$ in (278) and using $(aab) = a$, we get $(dea) = (dae)$. This together with (7.3) and (278) implies (7.4) and (7.5).

9. OTHER SYSTEMS OF AXIOMS

It has also been proved by the author of this thesis that (7.2) and

(7.8) $$(a0I) = a$$

together with one of

$$(1) \qquad ((abc)de) = (b(cde)(ade))$$

$$(2) \qquad = (b(ced)(aed))$$

$$(3) \qquad = (b(aed)(ced))$$

$$(4) \qquad = (c(ade)(bde))$$

$$(5) \qquad = (c(aed)(bed))$$

$$(6) \qquad = (c(bed)(aed))$$

and also (7.2) and

$$(7.9) \qquad (OIa) = a$$

together with one of

$$(7) \qquad (de(abc)) = ((dec)(dea)b)$$

$$(8) \qquad = ((edc)(eda)b)$$

$$(9) \qquad = ((eda)(edc)b)$$

$$(10) \qquad = ((deb)(dec)a)$$

$$(11) \qquad = ((edb)(edc)a)$$

$$(12) \qquad = ((edc)(edb)a)$$

form a set of three independent axioms for a distributive lattice

with 0 and I, the lattice operations defined by (7.6). In the following, the proofs are given for only two of these twelve sets. (It is easily verified that the identities (1) to (12) are exactly the identities of types (i) and (ii) defined on page 37.

THEOREM 9.1. The set of the axioms

$$(7.2) \qquad (aba) = a$$

$$(7.8) \qquad (a0I) = a$$

$$(4) \qquad ((abc)de) = (c(ade)(bde))$$

as well as the set of the axioms

$$(7.2) \qquad (aba) = a$$

$$(7.9) \qquad (0Ia) = a$$

$$(8) \qquad (de(abc)) = ((edc)(eda)b)$$

defines a distributive lattice with 0 and I, the lattice operations defined by (7.6).

Proof. Assume (7.2), (7.8) and (4). Then by (7.8), (4),

(7.8),

$$\begin{aligned}(abc) &= ((abc)0I)\\ &= (c(a0I)(b0I))\\ &= (cab).\end{aligned}$$

This implies that

$$\begin{aligned}(abc) = (cab) &= ((cac)ab)\\ &= (c(cab)a)\\ &= ((abc)ac)\\ &= (ca(bac))\\ &= ((bac)ca)\\ &= ((acb)ca) = (bac).\end{aligned}$$

Therefore (abc) is invariant under all permutations of a, b and c, and this together with (7.8) and (4) implies (7.1) and (7.3) to (7.5).

Again assume (7.2), (7.9) and (8). By (7.2) and (8), with $a = c = d = e$, $(aab) = a$, which implies that $(dea) = (eda)$ by (8) with $a = b = c$. This together with (7.9) and (8) with $d = 0$, $e = I$ implies that $(abc) = (cab)$ and therefore the operation (abc) is completely commutative, and (7.1), (7.3) to (7.5) hold. Therefore (7.2), (7.9) and (8) define a distributive lattice with 0 and I with the

lattice operations defined by (7.6).

All the twelve sets of axioms are independent by A_8 to A_{10}.

The author of this thesis has proved that (7.2), (7.3), (7.8) and one of

(13) $$((abc)de) = (a(cde)(bde))$$

(14) $$= (b(ade)(cde))$$

(15) $$= (c(bde)(ade));$$

(7.2), (7.3), (7.9) and one of

(16) $$(de(abc)) = ((dec)(deb)a)$$

(17) $$= ((dea)(dec)b)$$

(18) $$= ((deb)(dea)c);$$

(7.1), (7.3), $(baa) = a$ and one of

(19) $$((abc)de) = (a(bed)(ced))$$

(20) $$= (a(ced)(bed))$$

and, lastly, (7.1), (7.3), (aab) = a and one of

(21) $$(de(abc)) = ((eda)(edb)c)$$

(22) $$= ((edb)(eda)c)$$

form sets of four axioms for a distributive lattice with 0 and I with the lattice operations defined by (7.6). The proofs are omitted. The identities (13) to (22) are exactly the identities of type (iii) defined on page 37.

Finally one can easily see that each identity belonging to group G_4 together with (7.1) to (7.4) forms a set of five postulates for a distributive lattice with 0 and I with the lattice operations defined by (7.6).

In the following it will be shown how, in each set of axioms, for a distributive lattice with 0 and I, considered in this and the preceding section, the number of postulates can be reduced by one by combining or permuting or by both so that the new axioms are independent.

THEOREM 9.2. Each of the following fifteen sets of identities is a set of axioms for a distributive lattice with 0 and I, the lattice operations defined by (7.6):

(9.2.1) $$(a(0bI)a) = a$$

and one of

(1) $(0(d(abc)e)I) = ((dbe)c(dae))$

(2) $(d(abc)e) = (0((dbe)c(dae))I)$

(3) $((0dI)(abc)(0eI)) = ((dbe)c(dae))$

(4) $(d(abc)e) = ((0(dbe)I)c(0(dae)I))$

(5) $((0dI)(0(abc)I)(0eI)) = ((dbe)c(dae))$

(6) $(d(abc)e) = ((0(dbe)I)(0cI)(0(dae)I))$;

(7.2) and one of

(7) $(0(d(abc)e)I) = ((dbe)c(dae))$

(8) $((0dI)(abc)(0eI)) = ((dbe)c(dae))$

(9) $((0dI)(0(abc)I)(0eI)) = ((dbe)c(dae))$

(10) $(d(abc)e) = (0((dbe)c(dae))I)$

(11) $(d(abc)e) = ((0(dbe)I)c(0(dae)I))$

(12) $(d(abc)e) = ((0(dbe)I)(0cI)(0(dae)I))$;

and

(66) $(d(abc)e) = ((dbe)c(dae))$

and one of

(13) $$(O(aba)I) = a$$

(14) $$(O(baa)I) = a$$

(15) $$(O(aab)I) = a.$$

Proof. Let (*) be any one of (1) to (6). Assume (9.2.1) and (*). Replace each of a, b and c by (OaI) and d as well as e by a in (*) and use (9.2.1). Then we obtain (7.1). (7.1) and (9.2.1) imply (7.2). (7.1) and (*) imply (66).

Now put $d = O$ and $e = I$ in (66) and use (7.1) to get

(7.3) $$(abc) = (bca).$$

By (7.2) (7.3), (66) (7.3), (66) (7.2) (7.3), (7.3), (66) (7.2) (7.3),

$$\begin{aligned}(bca) &= ((abb)c(aab)) \\ &= (a(cab)b) \\ &= (ab(acb)) \\ &= (b(bac)a) \\ &= (acb).\end{aligned}$$

Hence (7.4) holds. (7.3), (7.4) and (66) imply (7.5).

In the next six cases, it is sufficient to prove (7.1) and (66).

Let Θ be any one of (7) to (12). Assume (7.2) and Θ. Put $a = b = c = d = e$ in Θ and use (7.2) to get (7.1). Then (7.1) and Θ imply (66).

In the last three cases, it is sufficient to prove (7.1) and (7.2).

Assume (66) and (13). Then, by (13), (66)(13),

$$(9.2.1) \qquad \begin{aligned} (aba) &= (O((aba)(cbc)(aba))I) \\ &= (c(aba)a). \end{aligned}$$

But by (13), (9.2.1), (13) (66),

$$\begin{aligned} a &= (O(aba)I) \\ &= (O((cbc)(aba)a)I) \\ &= (aac). \end{aligned}$$

Hence
$$\begin{aligned} (aba) &= ((aaa)b(aaa)) \\ &= (a(aab)a) = (aaa) = a. \end{aligned}$$

Hence (7.2) holds. (13) and (7.2) imply (7.1).

Let (14) and (66) hold. Then, by (14), (66) (14),

$$(9.2.2) \qquad (baa) = (0((bcc)(baa)(baa))I)$$
$$= (a(baa)c).$$

Again by (14), (9.2.2), (66) (14),

$$(9.2.3) \qquad (baa) = (0(b(baa)(baa))I)$$
$$= (0((baa)(b(baa)(baa))c)I)$$
$$= ((baa)ca).$$

But by (14), (9.2.3), (66) (14),

$$a = (0(baa)I)$$
$$= (0((baa)(bcc)a)I)$$
$$= (caa),$$

which together with (14) implies (7.1). We know that (7.1) and (66) imply (7.3). (7.3) and the identity $a = (caa)$ imply (7.2).

Lastly, assume (15) and (66). Then, by (15), (66), (15),

$$(aba) = ((0(aab)I)b(0(aab)I))$$
$$= (0((aab)(aab)b)I)$$
$$= (aab).$$

Therefore $(O(aba)I) = a$, and this case reduces to the case of (13). This completes the proof of the theorem.

Each of the first fourteen sets of axioms listed in this theorem is independent by A_8 and A_9. The set of (66) and (15) is independent by A_5 and A_8.

By Theorem 9.2, the set of the axioms (7.1), (7.2) and (66) of the previous section is reduced to a set of two independent axioms in fifteen different ways.

CHAPTER IV

POSTULATES FOR LATTICES AND SEMILATTICES

10. INTRODUCTION

As it is well known, a lattice is an algebraic system $\mathcal{A}$ consisting of a set A and two binary operations, $\vee$ and $\wedge$, on A which satisfy the identities

L_{10} $$(a \vee b) \vee c = a \vee (b \vee c)$$

L_{20} $$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

L_{30} $$a \vee (a \wedge b) = a$$

L_{40} $$a \wedge (a \vee b) = a$$

L_{50} $$a \vee b = b \vee a$$

(these six identities imply the idempotent laws for $\vee$ and $\wedge$)

L_{60} $$a \wedge b = b \wedge a$$

and a semilattice is an algebraic system $\mathcal{S}$ consisitng of a set S and a binary operation $\circ$ on S which satisfies the identities

S_{10} $$a \circ a = a$$

S_{20} $$a \circ b = b \circ a$$

S_{30} $$(a \circ b) \circ c = a \circ (b \circ c).$$

A. Petcu [8] considered the nine variants of L_{10}, the nine variants of L_{20}, the three variants of L_{30}, and the three variants of L_{40} which arise from L_{10}, L_{20}, L_{30} and L_{40}, respectively, by permuting and grouping a, b and c (in the cases of L_{30} and L_{40}: a and b) in all possible ways.

The variants of L_{10} considered by A. Petcu are:

L_{11} $$(a \vee b) \vee c = a \vee (c \vee b)$$

L_{12} $$(a \vee b) \vee c = b \vee (a \vee c)$$

L_{13} $$(a \vee b) \vee c = b \vee (c \vee a)$$

L_{14} $$(a \vee b) \vee c = (b \vee c) \vee a$$

L_{15} $$(a \vee b) \vee c = (c \vee b) \vee a$$

L_{16} $$(a \vee b) \vee c = (a \vee c) \vee b$$

L_{17} $$a \vee (b \vee c) = b \vee (a \vee c)$$

L_{18} $$a \vee (b \vee c) = c \vee (a \vee b)$$

L_{19} $a \vee (b \vee c) = c \vee (b \vee a).$

The corresponding variants of L_{20} are:

L_{21} $(a \wedge b) \wedge c = a \wedge (c \wedge b)$

L_{22} $(a \wedge b) \wedge c = b \wedge (a \wedge c)$

L_{23} $(a \wedge b) \wedge c = b \wedge (c \wedge a)$

L_{24} $(a \wedge b) \wedge c = (b \wedge c) \wedge a$

L_{25} $(a \wedge b) \wedge c = (c \wedge b) \wedge a$

L_{26} $(a \wedge b) \wedge c = (a \wedge c) \wedge b$

L_{27} $a \wedge (b \wedge c) = b \wedge (a \wedge c)$

L_{28} $a \wedge (b \wedge c) = c \wedge (a \wedge b)$

L_{29} $a \wedge (b \wedge c) = c \wedge (b \wedge a).$

The variants of L_{30} considered by A. Petcu are

L_{31} $a \vee (b \wedge a) = a$

L_{32} $(a \wedge b) \vee a = a$

L_{33} $(b \wedge a) \vee a = a.$

The corresponding variants of L_{40} are:

L_{41} $\quad a \wedge (b \vee a) = a$

L_{42} $\quad (a \vee b) \wedge a = a$

L_{43} $\quad (b \vee a) \wedge a = a.$

Let

$$\sigma_{ijk\ell} = \{L_{1i}, L_{2j}, L_{3k}, L_{4\ell}\}$$

for $\quad i, j = 0,\ldots,9; \quad k, \ell = 0,1,2,3.$

There are $10 \times 10 \times 4 \times 4 = 1600$ sets of this kind. We shall be interested in the question which of these sets are sets of axioms for a lattice.

Further, let us consider some combinations of L_{1i} and $L_{4\ell}$ and of L_{2j} and L_{3k}. Let, for $i, j = 0,\ldots,9$; $k, \ell = 0,1,2,3$,

$L^1_{1i;4\ell}$	$L^2_{1i;4\ell}$	$L^1_{2j;3k}$	$L^2_{2j;3k}$

be the identity arising from

$L_{4\ell}$	L_{3k}

by replacing b by t and a by the left side of

L_{1i}	L_{2j}

on the

left	right	left	right

side of

$L_{4\ell}$	L_{3k}

and replacing a by the right side of

L_{1i}	L_{2j}

on the

right	left	right	left

side of

$L_{4\ell}$	L_{3k}

For example, $L^1_{12;41}$ is the following identity:

$$((a \vee b) \vee c) \wedge (t \vee ((a \vee b) \vee c)) = b \vee (a \vee c).$$

If $i \in \{5,6,7,9\}$, $L^1_{1i;4\ell}$ and $L^2_{1i;4\ell}$ differ only in the notation and are, therefore, equivalent. If $j \in \{5,6,7,9\}$, $L^1_{2j;3k}$ and $L^2_{2j;3k}$ are equivalent for the same reason. Therefore, it is sufficient to consider the elements of

$$\{L^{\alpha}_{1i;4\ell} \mid i = 0,1,2,3,4,8;\ \ell = 0,1,2,3;\ \alpha = 1,2\}$$
$$\cup \{L^1_{1i;4\ell} \mid i = 5,6,7,9;\ \ell = 0,1,2,3\}$$

and the elements of

$$\{L^{\beta}_{2j;3k} \mid j = 0,1,2,3,4,8;\ k = 0,1,2,3;\ \beta = 1,2\}$$
$$\cup \{L^1_{2j;3k} \mid j = 5,6,7,9;\ k = 0,1,2,3\}.$$

The number of elements of each of these two sets is $6 \times 4 \times 2 + 4 \times 4 = 64$.

Let I^1 be the identity $a \vee a = a$, and let I^2 be the

identity $a \wedge a = a$.

Let

$$\sigma^{\alpha\beta\gamma}_{ijk\ell} = \{L^{\alpha}_{1i;4\ell}, L^{\beta}_{2j;3k}, I^{\gamma}\}$$

for $i, j = 0,\ldots,9$; $k, \ell = 0,1,2,3$; $\alpha, \beta, \gamma = 1,2$. When the $\sigma^{\alpha\beta\gamma}_{ijk\ell}$ with $i \in \{5,6,7,9\}$ and $\alpha = 2$ or $j \in \{5,6,7,9\}$ and $\beta = 2$ are disregarded, the number of all these sets of three identities is $64 \times 64 \times 2 = 8192$. We shall be interested in the question which of these 8192 sets of three identities are sets of axioms for a lattice.

A. Petcu [8] disregarded one of $L^{\alpha}_{1i;4\ell}$, $\alpha = 1,2$, even if $i \notin \{5,6,7,9\}$ and one of $L^{\beta}_{2j;3k}$, $\beta = 1,2$ even if $j \notin \{5,6,7,9\}$. Accordingly he considered only 3200 of these 8192 sets of identities.

It should be observed that none of the sets of three identities and none of the sets of four identities considered in this section includes L_{50} or L_{60}.

A. Petcu proved the following two theorems in [8].

THEOREM 1. $\sigma_{ijk\ell}$ as well as its dual is a set of lattice axioms if

$$i \in \{2,3\}, \quad j \in \{4,5\}, \quad k = 1, \ \ell = 0$$

or

$$i \in \{2,3\}, \quad j \in \{8,9\}, \quad k = 0, \ \ell = 2$$

or

$$i \in \{1,3\}, \quad j \in \{4,5\}, \quad k = 3, \ \ell = 1$$

or

$$i \in \{1,3\}, \quad j \in \{8,9\}, \quad k = 2, \ \ell = 3.$$

THEOREM 2. $\sigma^{11\gamma}_{ijk\ell}$ as well as its dual is a set of lattice axioms if $\gamma \in \{1,2\}$, and

$$i \in \{2,3\}, \quad j \in \{4,5\}, \quad k = 1, \ \ell = 0$$

or

$$i \in \{2,3\}, \quad j \in \{8,9\}, \quad k = 0, \ \ell = 2$$

or

$$i \in \{1,3\}, \quad j \in \{4,5\}, \quad k = 3, \ \ell = 1$$

or

$$i \in \{1,3\}, \quad j \in \{8,9\}, \quad k = 2, \quad \ell = 3$$

or

$$i \in \{8,9\}, \quad j \in \{4,5\}, \quad k = 3, \quad \ell = 0$$

or

$$i, j \in \{4,5\}, \quad k, \ell = 1$$

or

$$i, j \in \{8,9\}, \quad k, \ell = 2.$$

He also proved that the sets $\sigma_{ijk\ell}$, $\sigma_{ijk\ell}^{11\gamma}$, $i, j = 0,1,\ldots,9$; $k, \ell = 0,1,2,3$; $\gamma = 1,2$ are all independent. On the other hand, he stated 557 sets $\sigma_{ijk\ell}$ and proved that none of these sets is a set of lattice axioms.

The natural question that now arises is "what can we say about the remaining sets of three and four identities?"

Here we shall answer this question partly by stating 112 further sets of four axioms which define a lattice, 48 further sets $\sigma_{ijk\ell}$ which do not define a lattice, and 1128 sets of three axioms which define a lattice.

D.H. Potts [9] reduced the three axioms S_{1o} to S_{30} for a semilattice to the two identities S_{10} and

S_{40} $$((a o b) o (c o d)) o (e o f) = (b o a) o ((d o c) o (f o e)),$$

the latter in six variables. In this thesis, it is shown that a semilattice can be defined without postulating commutativity of o by considering two variants of S_{30} which are much simpler than S_{40} and involve only three variables.

11. POSTULATES

The two variants of S_{30} we are going to consider are

S_{31} $$(a o b) o c = (b o c) o a$$

and

S_{32} $$a o (b o c) = c o (a o b).$$

The following theorem gives two sets of two postulates for a semilattice.

THEOREM 11.1.*) Let S be an algebraic system consisting of a set S and a binary operation o on S satisfying the identities

S_{10} $$a o a = a$$

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*) This theorem was already proved by L. Byrne in his paper "Two Brief Formulations of Boolean Algebra", Bull. Am. Math. Soc., 52 (1946), 269-272

and one of

S_{31} $\qquad (aob) \circ c = (boc) \circ a$

S_{32} $\qquad a \circ (boc) = c \circ (aob).$

Then S is a semilattice.

Proof. It is sufficient to prove S_{20} and S_{30}. First we shall show that $\circ$ is commutative in either case.

Assume S_{10} and S_{31}. Put $c = a \circ b$ in S_{31} and use S_{10} and S_{31} to get

$$\begin{aligned} (aob) \circ (aob) &= (bo(aob)) \circ a, \\ a \circ b &= ((aob)oa) \circ b \\ &= ((aoa)ob) \, ob \\ &= (bob) \circ a \\ &= b \circ a. \end{aligned}$$

Secondly, assume S_{10} and S_{32}. Put $a = b \circ c$ in S_{32} and use S_{10} and S_{32} to get $b \circ c = c \circ b$.

Thus, in either case, the identities imply S_{20}. Then S_{30} follows from S_{20} and S_{31} or from S_{20} and S_{32}.

Let us now consider some independence examples.

A_{11}. Consider the set $B = \{u,v\}$, $u \neq v$ and define on it a binary operation o_1 by requiring that $a \; o_1 \; b = a$ if a and b are any elements of $\{u,v\}$. Then S_{10} is satisfied but none of S_{31} and S_{32} is valid.

A_{12}. Define a binary operation o_2 on the set B considered in A_{11} by requiring that $a \; o_2 \; b = u$ for $a \in \{u,v\}$, $b \in \{u,v\}$. Then S_{31} and S_{32} are valid while S_{10} fails to hold.

By A_{11} and A_{12}, the two sets of axioms considered in Theorem 11.1 are independent.

In the following we shall deal with sets of the forms $\sigma_{ijk\ell}$ and $\sigma_{ijk\ell}^{\alpha\beta\gamma}$.

LEMMA 1. Let k and ℓ be elements of $\{0,1,2,3\}$ such that $(k,\ell) \neq (0,3),(3.0),(1,1),(2,2)$. Let L_{3k} and $L_{4\ell}$ hold. Then I^1 and I^2 hold.

Proof. This is well-known for the case that $k = \ell = 0$. A proof can, e.g., be found on p.34 of [13]. For each of the other cases, the proof can be given in a similar way. Let us, e.g., give the proof for the case that $k = 0$ and $\ell = 1$. Put $b = b' \vee a$ in L_{30} and use L_{41} to get

$$a \vee (a \wedge (b' \vee a)) = a$$

and

$$I^1 \quad a \vee a = a.$$

Then I^1 together with L_{41} with $b = a$ implies I^2.

LEMMA 2. Let the hypotheses of Lemma 1 be satisfied. Let

i	j

be an element of $\{4,8\}$. Let

L_{1i}	L_{2j}

hold. Then

$(A,\vee)$	$(A,\wedge)$

is a semilattice.

Proof. $\vee$ and $\wedge$ are idempotent by Lemma 1.

L_{14}	L_{18}	L_{24}	L_{28}

is

S_{31}	S_{32}	S_{31}	S_{32}

with

$0 = \vee$.	$0 = \wedge$.

By Theorem 11.1,

$(A,\vee)$	$(A,\wedge)$

is a semilattice.

The following theorem gives 48 sets of four postulates for lattices.

THEOREM 11.2. Let i and j be elements of $\{4,8\}$. Let k and ℓ be elements of $\{0,1,2,3\}$ such that $(k,\ell) \neq (0,3)$, $(3,0)$, $(1,1)$, $(2,2)$. Then $\sigma_{ijk\ell}$ is a set of lattice axioms.

Proof. Let L_{1i}, L_{2j}, L_{3k} and $L_{4\ell}$ hold. Then, by Lemma 2, $(A,\vee)$ and $(A,\wedge)$ are semilattices. Hence $\vee$ and $\wedge$ are commutative. Therefore L_{1i}, L_{2j}, L_{3k} and $L_{4\ell}$ imply L_{10}, L_{20}, L_{30} and L_{40}. Hence $(A,\vee,\wedge)$ is a lattice.

The next theorem gives another 64 new sets of four postulates for a lattice. First we shall prove the following four

lemmata.

LEMMA 3. Let $i \in \{4,8\}$, $j \in \{1,3\}$, $k \in \{1,3\}$ and $\ell \in \{2,3\}$. Let L_{1i}, L_{2j}, L_{3k} and $L_{4\ell}$ hold. Then L_{60} holds.

Proof. By Lemma 1, I^1 and I^2 hold, and by Lemma 2, L_{50} holds. Hence L_{42} implies L_{43}, and vice versa. Hence L_{42} and L_{43} hold in any case. Putting $a = a' \wedge b$ in L_{42} and L_{43}, we find that

$$((a'\wedge b)\vee b) \wedge (a'\wedge b) = a' \wedge b$$

and

$$(b\vee(a'\wedge b)) \wedge (a'\wedge b) = a' \wedge b.$$

By L_{33} or by L_{31},

$$b \wedge (a'\wedge b) = a' \wedge b.$$

By L_{21} and I^2, or by L_{23} and I^2,

$$b \wedge a' = a' \wedge b.$$

Hence L_{60} holds.

LEMMA 4. Let $i \in \{4,8\}$, $j \in \{2,3\}$, $k \in \{0,2\}$ and

$\ell \in \{0,1\}$. Let L_{1i}, L_{2j}, L_{3k} and $L_{4\ell}$ hold. Then L_{60} holds.

Proof. By Lemmata 1 and 2 I^1, I^2 and L_{50} hold. By L_{50}, L_{40} is equivalent to L_{41}. Hence L_{40} and L_{41} hold in any case. By putting $a = b \wedge a'$ in L_{40} and L_{41}, we find that

$$(b\wedge a') \wedge ((b\wedge a')\vee b) = b \wedge a'$$

and

$$(b\wedge a') \wedge (b\vee(b\wedge a')) = b \wedge a'.$$

By L_{32} or by L_{30},

$$(b\wedge a') \wedge b = b \wedge a'.$$

By L_{22} and I^2, or by L_{23} and I^2, this reduces to $a' \wedge b = b \wedge a'$, which proves L_{60}.

LEMMA 5. Let $i \in \{4,8\}$, $j = 5$, and $k = 1$ and $\ell = 0$ or $k = 3$ and $\ell = 1$.

Let L_{1i}, L_{2j}, L_{3k} and $L_{4\ell}$ hold. Then L_{60} holds.

Proof. For the same reason as in the proof of Lemma 4, both L_{40} and L_{41} hold. Putting $a = a' \wedge b$ in L_{40} and L_{41}, we find that

$$(a' \wedge b) \wedge ((a' \wedge b) \vee b) = a' \wedge b$$

and

$$(a' \wedge b) \wedge (b \vee (a' \wedge b)) = a' \wedge b.$$

Now by L_{33} or L_{31}, we obtain

$$(a' \wedge b) \wedge b = a' \wedge b.$$

By L_{25} and Lemma 1,

$$b \wedge a' = a' \wedge b.$$

Hence L_{60} holds.

Lemma 6. Let $i \in \{4,8\}$, $j = 9$, and

$$k = 0 \text{ and } \ell = 2$$

or

$$k = 2 \text{ and } \ell = 3.$$

Let L_{1i}, L_{2j}, L_{3k} and $L_{4\ell}$ hold. Then L_{60} holds.

Proof. By Lemma 2, both L_{42} and L_{43} hold. Putting $a = b \wedge a'$ in L_{42} and L_{43} we obtain

$$((b \wedge a') \vee b) \wedge (b \wedge a') = b \wedge a'$$

and

$$(b \vee (b \wedge a')) \wedge (b \wedge a') = b \wedge a'.$$

Then using L_{32} or L_{30}, we get

$$b \wedge (b \wedge a') = b \wedge a'.$$

By L_{29} and Lemma 1, this is transformed into

$$a' \wedge b = b \wedge a'.$$

Therefore L_{60} holds.

THEOREM 11.3. Let $i = 4$ or $i = 8$. Let

$$j \in \{1,3\}, \quad k \in \{1,3\}, \quad \ell \in \{2,3\}$$

or

$$j \in \{2,3\}, \quad k \in \{0,2\}, \quad \ell \in \{0,1\}$$

or

$$j = 5, \quad k = 1, \quad \ell = 0$$

or

$$j = 5, \quad k = 3, \quad \ell = 1$$

or

$$j = 9, \quad k = 0, \quad \ell = 2$$

or

$$j = 9, \quad k = 2, \quad \ell = 3.$$

Then $\sigma_{ijk\ell}$ as well as its dual is a set of lattice axioms.

Proof. Let L_{1i}, L_{2j}, L_{3k} and $L_{4\ell}$ hold. Then it follows from Lemmata 2 to 6 that L_{50} and L_{60} hold. Hence L_{1i}, L_{2j}, L_{3k} and $L_{4\ell}$ imply L_{10} to L_{40}. Hence $(A,\vee,\wedge)$ is a lattice.

By the duality principle, the dual of $\sigma_{ijk\ell}$ also defines a lattice.

This theorem states 40 quadruplets (i,j,k,ℓ) for which $\sigma_{ijk\ell}$ as well as its dual is a set of lattice axioms. But for 8 of these cases, this is already contained in Petcu's Theorem 1. These are the following cases:

(a) $i = 4$ (i) $j \in \{1,3\}$, $k = 1$, $\ell = 3$

(ii) $j \in \{2,3\}$, $k = 0$, $\ell = 1$

(b) $i = 8$ (i) $j \in \{1,3\}$, $k = 3$, $\ell = 2$

(ii) $j \in \{2,3\}$, $k = 2$, $\ell = 0$.

Hence only the remaining 32 cases are new, and Theorem 11.3 states only 64 new sets of four lattice axioms.

Thus by Theorems 11.2 and 11.3 we get 112 new sets of four lattice axioms.

Each set of lattice axioms involved in Theorems 11.2 and 11.3 is independent by propositions $1^0, 1^1, 2^0$ and 2^1 which occur on p. 343 of [8].

The following theorem gives 48 new sets of four identities which do not define a lattice.

THEOREM 11.4. Let $i = 0$, $j \in \{0,1,\ldots,9\}$, $k \in \{0,1,2,3\}$ and $\ell \in \{1,3\}$. Then $\sigma_{ijk\ell}$ or its dual does not define a lattice.

Proof. Consider the set $A = \{u,v,w\}$ of three different elements and define on it two binary operations, $\vee$ and $\wedge$, by

$$a \vee b = \begin{cases} a & \text{if } b = v \\ b & \text{otherwise} \end{cases}$$

and

$$a \wedge b = \begin{cases} a & \text{if } a = b \\ v & \text{otherwise,} \end{cases}$$

where the letters a and b denote any elements of the set A. Now either side of L_{10} is equal to $a \vee b$ if $c = v$ and equal to c if $c \neq v$. Hence L_{10} holds. Either side of L_{2j} is equal to a if $a = b = c$ and equal to v otherwise. Hence L_{2j} holds. It is easily seen that L_{3k} and $L_{4\ell}$ are also satisfied. But v is non-commutative by definition. Hence $(A,\vee,\wedge)$ is not a lattice. Therefore $\sigma_{ijk\ell}$ does not define a lattice.

By the duality principle, the dual of $\sigma_{ijk\ell}$ does not define a lattice.

Theorem 11.4 states 156 cases in which $\sigma_{ijk\ell}$ does not define a lattice. For 108 of those 156 cases, this has already been proved by A. Petcu. But it is new in the following 24 cases and their duals:

(i) $i = 0$, $j \in \{3,4,5,8,9\}$, $k \in \{0,1\}$, $\ell \in \{1,3\}$;

(ii) $i = 0$, $j \in \{2,7\}$, $k = 0$, $\ell = 1$;

(iii) $i = 0$, $j \in \{1,6\}$, $k = 1$, $\ell = 3$.

Now we shall consider sets of three identities of type

$\sigma^{\alpha\beta\gamma}_{ijk\ell}$.

The following theorem gives 1152 such sets of lattice axioms.

THEOREM 11.5. Let $i \varepsilon \{4,8\}$ and $\alpha, \beta, \gamma \varepsilon \{1,2\}$. Let

(a) $j \varepsilon \{4,8\}$, k, $\ell \varepsilon \{0,1,2,3\}$

or

(b) $j \varepsilon \{1,3\}$, $k \varepsilon \{1,3\}$, $\ell \varepsilon \{2,3\}$

or

(c) $j \varepsilon \{2,3\}$, $k \varepsilon \{0,2\}$, $\ell \varepsilon \{0,1\}$

or

(d) $j = 5$, $k \varepsilon \{1,3\}$, $\ell \varepsilon \{0,1\}$

or

(e) $j = 9$, $k \varepsilon \{0,2\}$, $\ell \varepsilon \{2,3\}$.

Then $\sigma^{\alpha\beta\gamma}_{ijk\ell}$ as well as its dual is a set of lattice axioms.

Proof. Let $L^{\alpha}_{1i;4\ell}$, $L^{\beta}_{2j,3k}$ and I^{γ} hold. Let, e.g., $\gamma = 1$. Then, putting $a = b = c$ in $L^{\alpha}_{1i;4\ell}$ and using I^1, we obtain $L_{4\ell}$, and $L_{4\ell}$ together with $L^{\alpha}_{1i;4\ell}$ and I^1 implies L_{1i} and I^2. Similarly, if $\gamma = 2$, L_{3k}, L_{2j} and I^1 hold. Hence L_{1i}, L_{2j}, L_{3k}, $L_{4\ell}$, I^1 and I^2 hold in both cases. By Theorem 11.1, with $o = \vee$, L_{1i} and I^1 imply that $(A,\vee)$ is a semilattice. Hence L_{50} holds.

In case (a), by Theorem 11.1, with $o = \wedge$, L_{2j} and I^2

imply L_{60}. In cases (b) and (c), L_{60} holds by Lemmata 3 and 4, respectively.

Suppose we have case

(d).	(e).

Then

L_{31} and L_{41}	L_{30} and L_{43}

hold. Putting

$a = a' \wedge b$	$a = b \wedge a'$

in

L_{41}	L_{43}

and using

L_{31}	L_{30}

we find that

$(a' \wedge b) \wedge b = a' \wedge b$	$b \wedge (b \wedge a') = b \wedge a'.$

By

L_{25}, $(b \wedge b) \wedge a' = a' \wedge b.$	L_{29}, $a' \wedge (b \wedge b) = b \wedge a'.$

By I^2,

$$b \wedge a' = a' \wedge b.$$

Hence L_{60} holds.

Thus L_{60} holds in all five cases, (a) to (e). L_{1i}, L_{2j}, L_{3k}, $L_{4\ell}$, L_{50} and L_{60} imply L_{10}, L_{20}, L_{30} and L_{40}. Hence $(A,\vee,\wedge)$ is a lattice.

By the duality principle, the dual of $\sigma_{ijk\ell}^{\alpha\beta\gamma}$ also defines a lattice.

Thus, as the set of the 512 cases of (a) is selfdual, Theorem 11.5 states only 1152 cases in which $\sigma_{ijk\ell}^{\alpha\beta\gamma}$ is a set of lattice axioms. But, for 56 of those 1152 cases, this has already been proved by A. Petcu. These are the following cases and their duals:

$$\alpha = \beta = 1, \quad \gamma \in \{1,2\},$$

(i) $i = 4, \quad j = 8, \quad k = 0, \quad \ell = 3$

(ii) $i = j = 4, \quad k = \ell = 1$

(iii) $i = j = 8, \quad k = \ell = 2$

(iv) $i = 4, \quad j \in \{1,3\}, \quad k = 1, \quad \ell = 3$

(v) $i = 4, \quad j \in \{2,3\}, \quad k = 0, \quad \ell = 1$

(vi) $i = 8, \quad j \in \{1,3\}, \quad k = 3, \quad \ell = 2$

(vii) $i = 8, \quad j \in \{2,3\}, \quad k = 2, \quad \ell = 0$

(viii) $i = 4, \quad j = 5, \quad k = \ell = 1$

(ix) $i = 8, \quad j = 5, \quad k = 3, \quad \ell = 0$

(x) $i = 4, \quad j = 9, \quad k = 0, \quad \ell = 3$

(xi) $i = 8, \quad j = 9, \quad k = \ell = 2.$

Hence only the remaining 1096 cases are new. Thus Theorem 11.5 gives 1096 new sets of three lattice axioms.

A. Petcu disregarded 168 sets of three identities in Theorem 2 of [8]. Only 32 of those sets are not contained in Theorem 11.5. They are the following cases and their duals:

$$\alpha = \beta = 2, \quad \gamma = 1,2 \quad \text{and}$$

(i) $i = 5, \quad j \in \{2,3\}, \quad k = 0, \quad \ell = 1$

(ii) $i = 9, \quad j \in \{2,3\}, \quad k = 2, \quad \ell = 0$

(iii) $i = 5, \quad j \in \{1,3\}, \quad k = 1, \quad \ell = 3$

(iv) $i = 9$, $j \in \{1,3\}$, $k = 3$, $\ell = 2$.

These 32 sets also define a lattice and the proof is contained in the proof of Theorem 2 of [8].

Thus we get $1096 + 32 = 1128$ new sets of three lattice axioms altogether.

Each of these 1128 sets of three axioms is independent by propositions 9, 10^0 and 10^1 of [8].

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